2. WITT'S DECOMPOSITION THEOREM

§2.1. Direct Sums

If U, V are quadratic spaces over F then **U ⊕ V** is the **direct sum** of the vector spaces U, V made into a quadratic space by defining

$$\langle(\mathbf{x}_1, \mathbf{x}_2) \mid (\mathbf{y}_1, \mathbf{y}_2)\rangle = \langle\mathbf{x}_1 \mid \mathbf{y}_1\rangle_U + \langle\mathbf{x}_2 \mid \mathbf{y}_2\rangle_V.$$

Example 1: In $\mathbb{C}_R \oplus \mathbb{C}_I$,

$$\begin{aligned}\langle(\mathbf{i}, \mathbf{i}) \mid (\mathbf{1}, \mathbf{i})\rangle &= \langle\mathbf{i} \mid \mathbf{1}\rangle_{\mathbb{C}_R} + \langle\mathbf{i} \mid \mathbf{i}\rangle_{\mathbb{C}_I} \\ &= \mathrm{Re}(\mathbf{i}) + \mathrm{Im}(-\mathbf{1}) \\ &= 0.\end{aligned}$$

Hence $(\mathbf{i}, \mathbf{i})$ and $(\mathbf{1}, \mathbf{i})$ are orthogonal.

If A, B are square matrices, $A \oplus B = \begin{pmatrix} A & O \\ O & B \end{pmatrix}$. The direct sum of two quadratic forms $\Sigma a_{ij} x_i x_j$ and $\Sigma b_{ij} x_i x_j$ is $\Sigma c_{ij} x_i x_j$ where $(c_{ij}) = (a_{ij}) \oplus (b_{ij})$.

Example 2: $(x_1^2 - x_2^2) \oplus x_1 x_2 = x_1^2 - x_2^2 + x_3 x_4$.

NOTE: ⊕ is associative and commutative up to isomorphism, congruence and equivalence.

In the following 1-1 correspondences direct sums correspond.

Congruence classes of SYMMETRIC MATRICES ↔ Equivalence classes of QUADRATIC FORMS ↔ Isometry classes of QUADRATIC SPACES

NOTE:
$\langle a_1, \dots, a_n\rangle \oplus \langle b_1, \dots, b_m\rangle \cong \langle a_1, \dots, a_n, b_1, \dots, b_m\rangle$.

Notation: *n*A is $A \oplus A \oplus \dots \oplus A$ where A is a matrix, quadratic form or quadratic space.

§2.2. Hyperbolic Spaces

A **hyperbolic plane** is a 2-dimensional quadratic space whose quadratic form is equivalent to $\langle 1, -1\rangle$. Equivalently a hyperbolic plane is equivalent to xy.

Notation: A hyperbolic plane is denoted by $\mathcal{H}$.

A quadratic space V is a **hyperbolic space** if
$V \cong n\mathrm{H}$ for some $n \in \mathbb{Z}$.
The dimension of a hyperbolic space is even.

Example 3: $\mathbb{C}_I$, $\mathbb{C}_R$ are hyperbolic planes. $\mathbb{C}_R \oplus \mathbb{C}_I$ is a hyperbolic space.

Theorem 1: A 2-dimensional quadratic space is a hyperbolic plane if and only if its determinant is equivalent to -1.

Proof: Suppose $ab \cong -1$. Then $ab = -k^2$ for some $k \in F^{\#}$.

Then $\langle a, b\rangle \cong \langle a, -k^2/a\rangle$

$$\begin{aligned}
&= ax^2 - \frac{k^2}{a} y^2 \\
&= \frac{a}{a^2} (a^2 x^2 - k^2 y^2) \\
&= a \left(\frac{ax - ky}{a}\right)\left(\frac{ax + ky}{a}\right) \\
&\cong axy \\
&\cong \mathrm{z}y \text{ (where } \mathrm{z} = ax\text{)} \\
&\cong \langle 1, -1\rangle.
\end{aligned}$$

The converse is obvious. ✋☺

§2.3. Isotropic and Regular Spaces

A quadratic space is **isotropic** if there exists $\mathbf{v} \neq 0$ such that $\langle \mathbf{v} \mid \mathbf{v}\rangle = 0$.

Example 4: $\mathcal{H}$ is isotropic.

The **radical** of a quadratic space is **rad V** $= V^{\perp}$.

A quadratic space is **regular** if rad V = 0, that is, if the corresponding matrix is non-singular or if the determinant of the corresponding quadratic form is non-zero.

Theorem 2: A quadratic space is either regular or isotropic.

Proof: If V is not regular there exists a non-zero vector $\mathbf{v} \in \text{rad } V$.

Hence $\langle \mathbf{v} \mid \mathbf{v} \rangle = 0$. ✋☺

Example 5:

(1) $\mathcal{H} = \langle 1, -1 \rangle$ is both isotropic and regular.

(2) $\langle 1, 0 \rangle$ is isotropic but not regular.

(3) $\langle 1, 1 \rangle$ is regular but not isotropic.

Theorem 3: If M is a regular subspace of the quadratic space V then $V = M \oplus M^{\perp}$.

Proof: $V = M + M^{\perp}$: Let $\mathbf{e}_1, \ldots, \mathbf{e}_r$ be an orthogonal basis of M and let $\mathbf{v} \in \mathbf{V}$.

If, for any i, $\langle \mathbf{e}_i \mid \mathbf{e}_i \rangle = 0$ then $\mathbf{e}_i \in \text{rad } M = 0$, since M is regular.

Hence $\langle \mathbf{e}_i \mid \mathbf{e}_i \rangle \neq 0$ for each i.

Let $$\mathbf{w} = \mathbf{v} - \sum_{i=1}^{r} \frac{\langle \mathbf{v} \mid \mathbf{e}_i \rangle}{\langle \mathbf{e}_i \mid \mathbf{e}_i \rangle} \mathbf{e}_i \, .$$

Then $\langle \mathbf{w} \mid \mathbf{e}_j \rangle = 0$ for all j.

Hence $\mathbf{w} \in M^{\perp}$ and so $\mathbf{v} \in M + M^{\perp}$.

$\mathbf{M \cap M^{\perp} = 0}$: since $M \cap M^{\perp} = \text{rad } M = 0$.

We now show that **$\mathbf{V = M \oplus M^{\perp}}$ as quadratic spaces:**

If $\mathbf{m}_1, \mathbf{m}_2 \in M$ and $\mathbf{n}_1, \mathbf{n}_2 \in M^{\perp}$ then

$$\begin{aligned}\langle \mathbf{m}_1 + \mathbf{n}_1 \mid \mathbf{m}_2 + \mathbf{n}_2\rangle &= \langle \mathbf{m}_1 \mid \mathbf{m}_2\rangle + \langle \mathbf{m}_1 \mid \mathbf{n}_2\rangle + \langle \mathbf{n}_1 \mid \mathbf{m}_2\rangle \\ &\quad + \langle \mathbf{n}_1 \mid \mathbf{n}_2\rangle \\ &= \langle \mathbf{m}_1 \mid \mathbf{m}_2\rangle + \langle \mathbf{n}_1 \mid \mathbf{n}_2\rangle.\end{aligned}$$ ✋☺

Theorem 4: V is regular and isotropic if and only if $V \cong H \oplus W$ where H is the hyperbolic space and W is regular.

Proof: Suppose V is regular and isotropic.
Let $\mathbf{x} \neq \mathbf{0}$ be such that $\langle \mathbf{x} \mid \mathbf{x}\rangle = 0$.
Since V is regular there exists $\mathbf{y} \in V$ such that $\langle \mathbf{x} \mid \mathbf{y}\rangle \neq 0$.
Let $H = \langle \mathbf{x}, \mathbf{y}\rangle$ the subspace spanned by $\mathbf{x}, \mathbf{y}$.
Then

$$\begin{aligned}\langle \lambda\mathbf{x} + \mathbf{y} \mid \lambda\mathbf{x} + \mathbf{y}\rangle &= \lambda^2\langle \mathbf{x} \mid \mathbf{x}\rangle + 2\lambda\langle \mathbf{x} \mid \mathbf{y}\rangle + \langle \mathbf{y} \mid \mathbf{y}\rangle \\ &= 2\lambda\langle \mathbf{x} \mid \mathbf{y}\rangle + \langle \mathbf{y} \mid \mathbf{y}\rangle.\end{aligned}$$

Let $\lambda_1 = \dfrac{1 - \langle \mathbf{y} \mid \mathbf{y}\rangle}{2\langle \mathbf{x} \mid \mathbf{y}\rangle}$ and $\lambda_2 = \dfrac{-1 - \langle \mathbf{y} \mid \mathbf{y}\rangle}{2\langle \mathbf{x} \mid \mathbf{y}\rangle}$.
Let $\mathbf{e}_1 = \lambda_1\mathbf{x} + \mathbf{y}$ and $e_2 = \lambda_2\mathbf{x} + \mathbf{y}$.
Then $\langle \mathbf{e}_1 \mid \mathbf{e}_1\rangle = 1$, $\langle \mathbf{e}_2 \mid \mathbf{e}_2\rangle = -1$ and $\langle \mathbf{e}_1 \mid \mathbf{e}_2\rangle = 0$.
So $\mathbf{e}_1, \mathbf{e}_2$ is an orthogonal basis for $\mathcal{H}$ relative to which the quadratic form is $\langle 1, -1\rangle$.
Hence H is a hyperbolic space.
Since H is regular, $V = \mathcal{H} \oplus \mathcal{H}^{\perp}$ by Theorem 3.
Let $W = \mathcal{H}^{\perp}$. Clearly $\mathcal{H}^{\perp}$ is regular. ✋☺

§2.4. Witt's Decomposition Theorem

Let **Z** be the 1-dimensional quadratic space associated with the quadratic form $0x^2$, that is $Z \cong \langle 0 \rangle$. In other words $Z = \langle \mathbf{x} \rangle$ where $\langle \mathbf{x} \mid \mathbf{x} \rangle = 0$.

Theorem 5: (Witt's Decomposition Theorem):
If V is a quadratic space then $V \cong mZ \oplus nH \oplus W$ for some regular non-isotropic quadratic space W.
Proof: $V = \text{rad } V \oplus K$ for some K:
Choose a basis for rad V (necessarily orthogonal) and an orthogonal basis for K, and combine. Then

- $\text{rad } V = mZ$ for some m
- K is regular
- $K \cong nH \oplus W$ for some n and some regular non-isotropic W by Theorem 4. ✋☺

Example 6: Over $\mathbb{Q}$:
$\langle 3, -5, 2, 0 \rangle \cong \langle 0 \rangle \oplus \langle 3, -5, 2 \rangle$ (regular and isotropic since $3.1^2 - 5{,}1^2 + 2.1^2 = 0$)
$\cong Z \oplus H \oplus \langle d \rangle$ for some d (by Theorem 4)
$\cong Z \oplus H \oplus \langle 30 \rangle$ by considering determinants.